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Azimuthal Projection with Three Standard Parallels

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1. Introduction

Developable surfaces ?
Secant surfaces or projections?
Generally not correct approach

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Snyder (1987): "... most azimuthal maps do not have standard parallels or standard meridians. Each map only has one standard point: the center (except for the stereographic projection, which may have a standard circle)".

These statements by Snyder are not correct.

There are azimuthal projections without standard parallels, but also those with one, two, three or more standard parallels!

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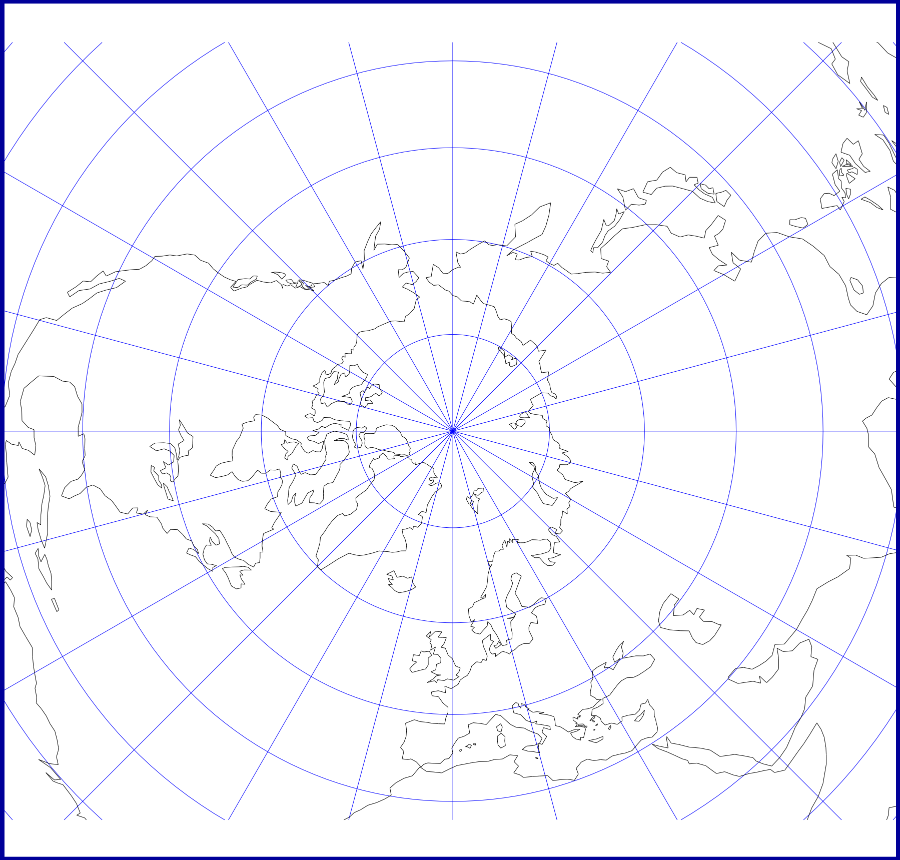
Azimuthal projection

$x = \rho \cos \delta$ $y = \rho \sin \delta$

$\rho = \rho(\varphi)$ $\delta = \lambda - \lambda_0$



φ ... latitude

λ ... longitude

λ_0 ... longitude of the central meridian of the mapped area

δ ... angle of intersecting meridian images in the plane of projection

ρ ... radius of a parallel in the plane of projection

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The distortion distribution is affected by the selection of standard parallels, i.e. parallels along which there are no distortions.

$$m = m(\varphi) = -\frac{d\rho}{R d\varphi}$$

$$n = n(\varphi) = \frac{\rho}{R\cos\varphi}$$

The condition for a parallel to be a standard parallel reads

$$m(\varphi) = n(\varphi) = 1$$

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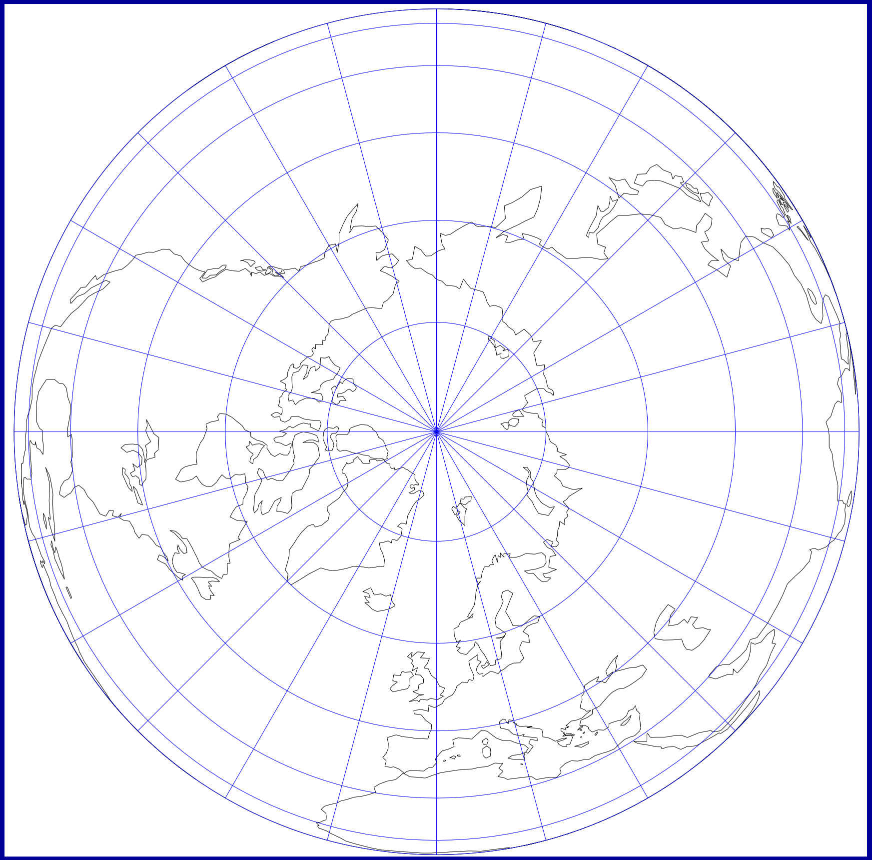
2. Equidistant Along Parallels, or Orthographic Projection

$$n(\varphi) = 1$$

$$\rho = R\cos\varphi$$



$$m(\varphi) = \sin\varphi$$

$$m(\varphi) = n(\varphi) = 1$$

$$\sin\varphi = 1$$

North Pole is the only point in the projection that is mapped without distortions

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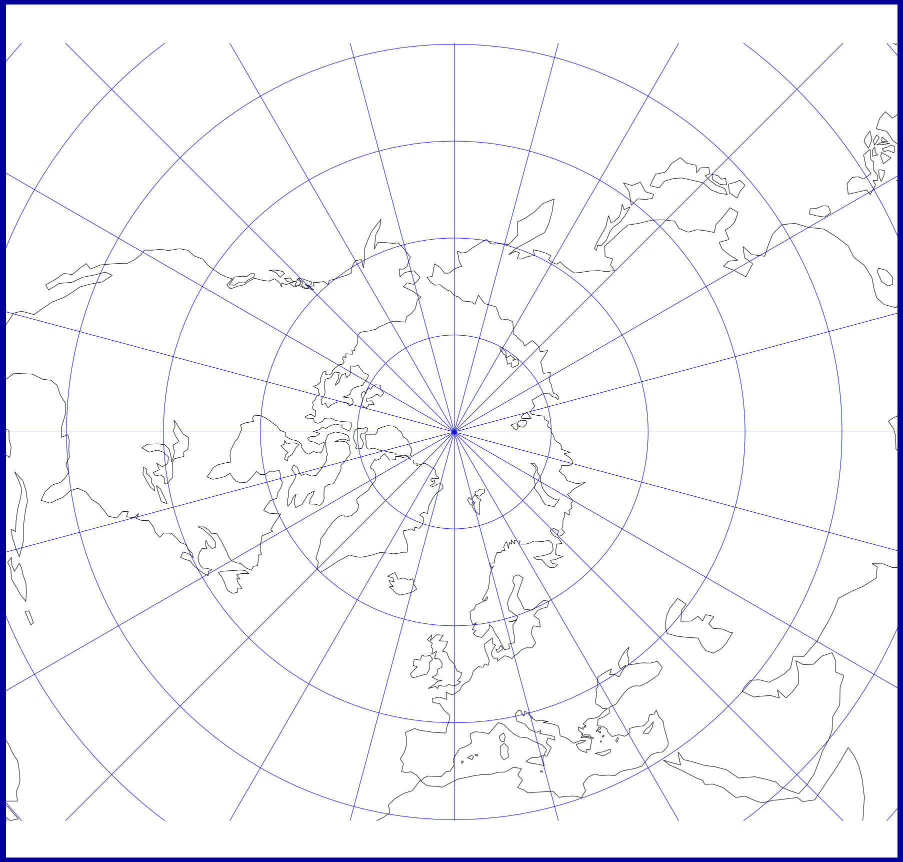
3. Equidistant Along Meridians, or Postel Projection

$$m(\varphi) = 1$$

$$-\frac{d\rho}{d\varphi} = R$$

$$\rho = K - R\varphi$$



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3. Equidistant Along Meridians, or Postel Projection (cont.)

$$n(\varphi) = \frac{K - R\varphi}{R\cos\varphi}$$

$$m(\varphi) = n(\varphi) = 1$$

$$\varphi + \cos\varphi = \frac{K}{R}$$

This equation will have a solution if, and only if the following condition is true

$$-\frac{\pi}{2}R \le K \le \frac{\pi}{2}R$$

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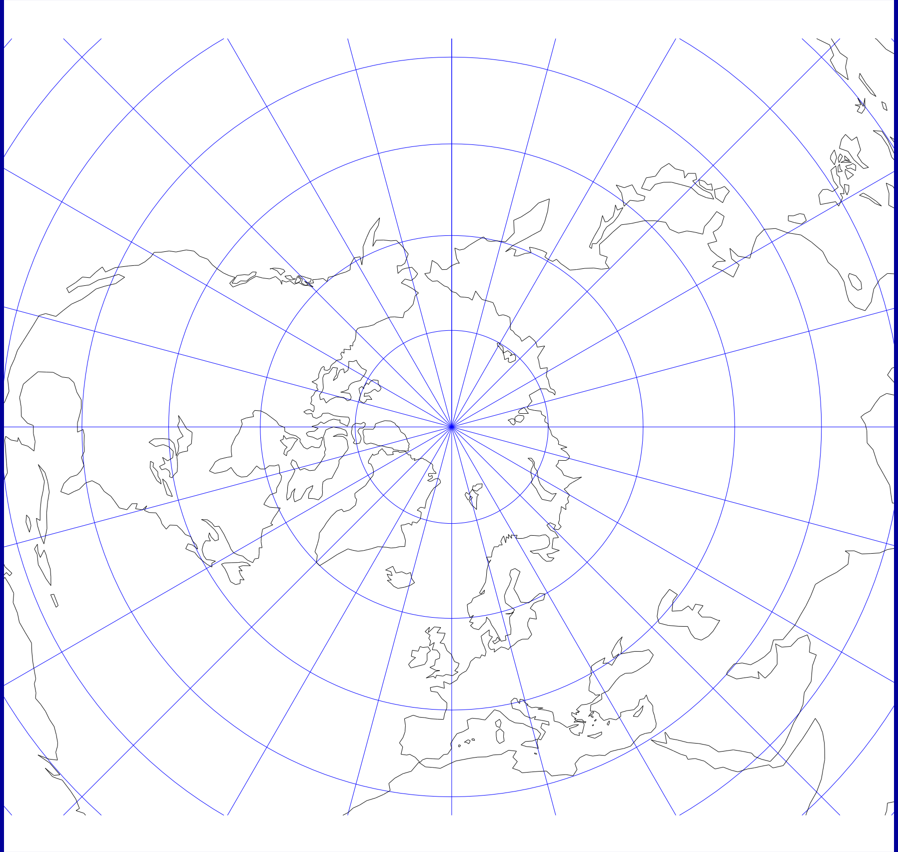
4. Equal-area, or Lambert Azimuthal Projection

$$m(\varphi)n(\varphi) = 1$$

$$-\frac{\rho}{\cos\varphi}\frac{d\rho}{d\varphi} = R^2$$

$$\rho = \sqrt{K - 2R^2 \sin\varphi}$$



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4. Equal-area, or Lambert Azimuthal Projection (cont.)

$$m = \frac{1}{n} = \frac{R\cos\varphi}{\sqrt{K - 2R^2\sin\varphi}}$$

$$m(\varphi) = n(\varphi) = 1$$

$$\sin^2\varphi - 2\sin\varphi + \frac{K}{R^2} - 1 = 0$$

There is always only one solution

$$\sin\varphi_0 = 1 - \sqrt{2 - \frac{K}{R^2}}$$

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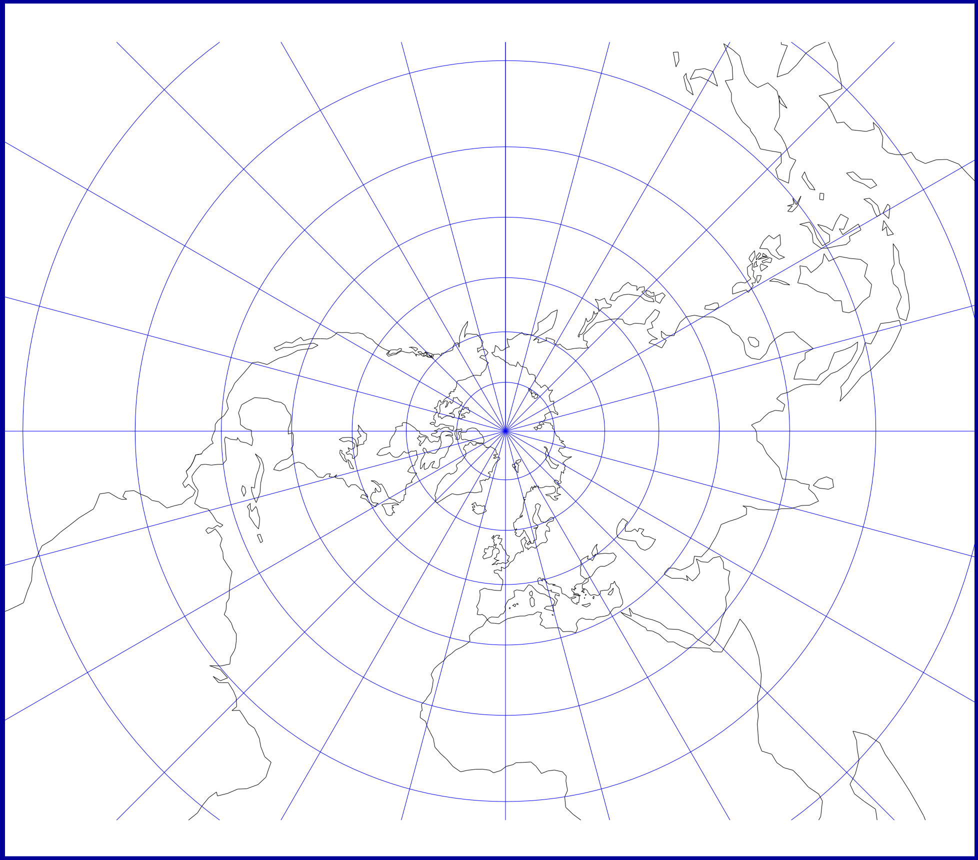
5. Conformal, or Stereographic Azimuthal Projection

$$m(\varphi) = n(\varphi)$$

$$-\frac{d\rho}{d\varphi} = \frac{\rho}{\cos\varphi}$$

$$\rho = K \tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)$$



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5. Conformal, or Stereographic Azimuthal Projection (cont.)

$$m = n = \frac{K}{2R\cos^2\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)} = \frac{K}{R(1+\sin\varphi)}$$

$$m(\varphi) = n(\varphi) = 1$$

$$\sin\varphi = \frac{K}{R} - 1$$

No solution, or one solution only

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6. Perspective Azimuthal Projection

$$\rho = \frac{LR\cos\varphi}{D + R\sin\varphi}$$

$$m(\varphi) = \frac{L(D\sin\varphi + R)}{(D + R\sin\varphi)^2}$$

$$n(\varphi) = \frac{L}{D + R\sin\varphi}$$

$$m(\varphi) = n(\varphi) = 1$$

The projection generally does not have standard parallels, except in the two cases:

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6. Perspective Azimuthal Projection (cont.)

The projection generally does not have standard parallels, except in the two cases defined by:

$$(D-R)(L-D-R)=0$$

$D=R$ stereographic projection

$L=D+R$ perspective azimuthal projection with the North Pole as a standard point only

7. Azimuthal Projection with Two Standard Parallels

There are azimuthal projections without standard points, with one standard point or with a standard parallel.

Are there azimuthal projections with more standard parallels, e.g. with two or three standard parallels?

?

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7. Azimuthal Projection with Two Standard Parallels (cont.)

We are searching for a function

$$\rho = \rho(\varphi)$$

which has to fulfill

$$m(\varphi) = n(\varphi) = 1$$

which means

$$m(\varphi) = -\frac{d\rho}{R d\varphi} = -\frac{\rho'}{R} = 1 \qquad n(\varphi) = \frac{\rho}{R\cos\varphi} = 1$$

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7. Azimuthal Projection with Two Standard Parallels (cont.)

$\varphi = \varphi_1$ $\varphi = \varphi_2$

$\rho(\varphi_1) = R\cos\varphi_1$ $\rho(\varphi_2) = R\cos\varphi_2$

$\rho'(\varphi_1) = -R$ $\rho'(\varphi_2) = -R$

We need a function $\rho = \rho(\varphi)$

passing through two given points in a given direction

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7. Azimuthal Projection with Two Standard Parallels (cont.)

The problem has an infinite number of solutions.

$$\rho = a\varphi^3 + b\varphi^2 + c\varphi + d$$

$$\begin{bmatrix} \varphi_1^3 & \varphi_1^2 & \varphi_1 & 1 \\ \varphi_2^3 & \varphi_2^2 & \varphi_2 & 1 \\ 3\varphi_1^2 & 2\varphi_1 & 1 & 0 \\ 3\varphi_2^2 & 2\varphi_2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} R\cos\varphi_1 \\ R\cos\varphi_2 \\ -R \\ -R \end{bmatrix}$$

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7. Azimuthal Projection with Two Standard Parallels (cont.)

Example

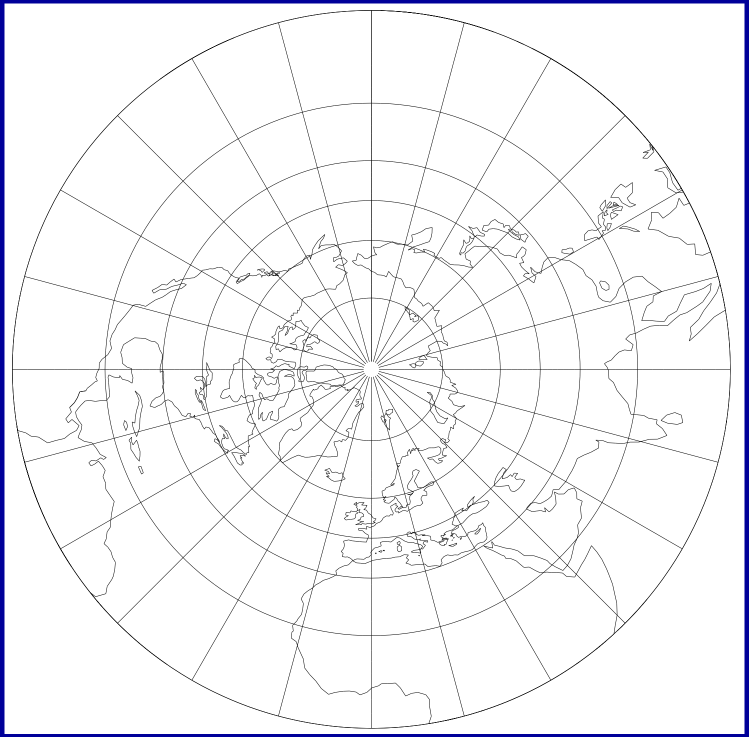
$R = 1$ $\varphi_1 = 15°\text{N}$ $\varphi_2 = 75°\text{N}$

$a = -0.59230$

$b = 1.39557$

$c = -1.60893$

$d = 1.30212$



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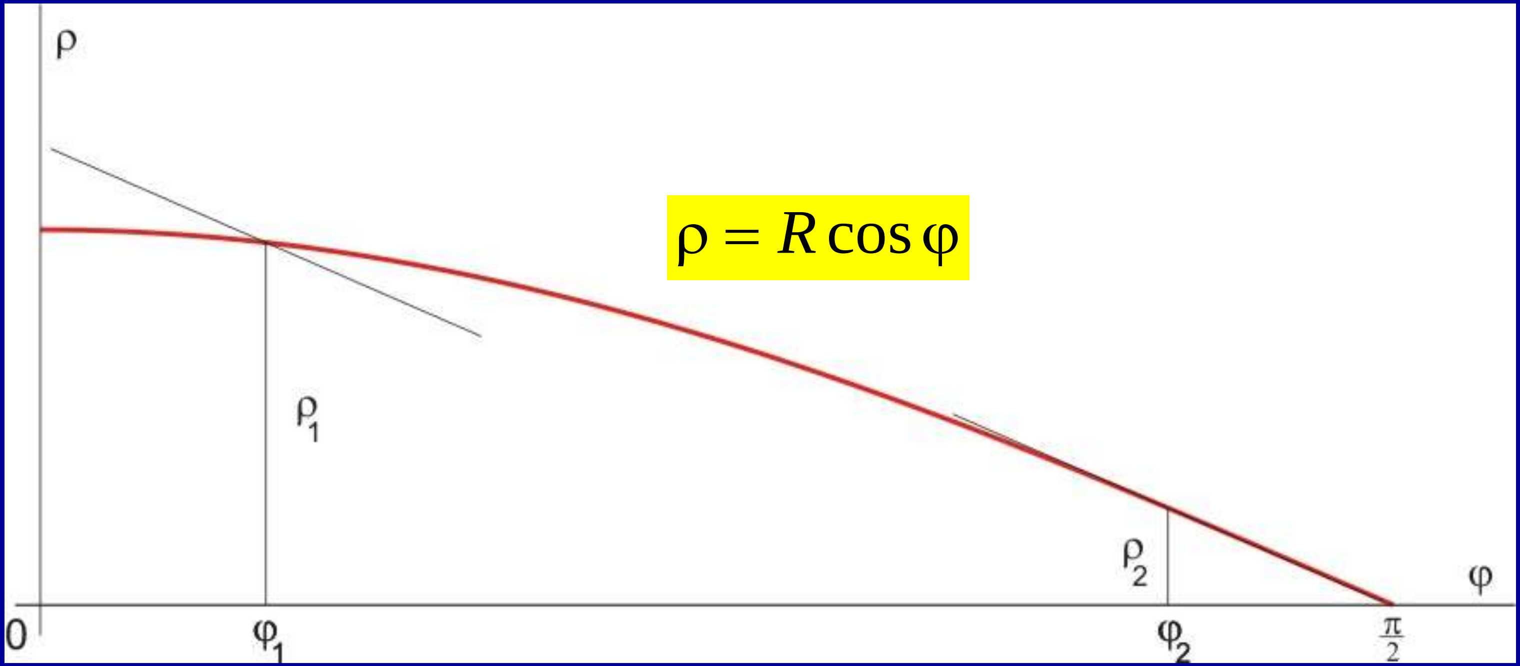
7. Azimuthal Projection with Two Standard Parallels (cont.)



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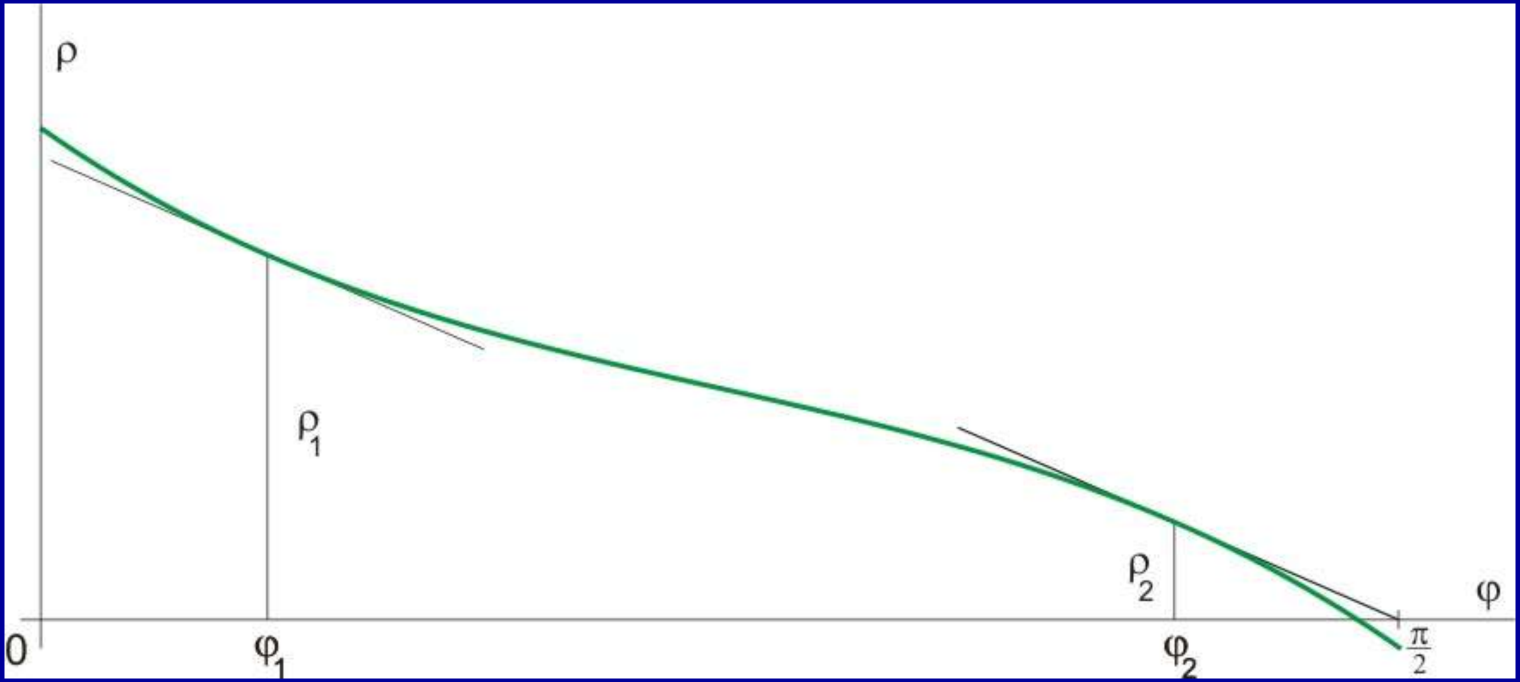
7. Azimuthal Projection with Two Standard Parallels (cont.)



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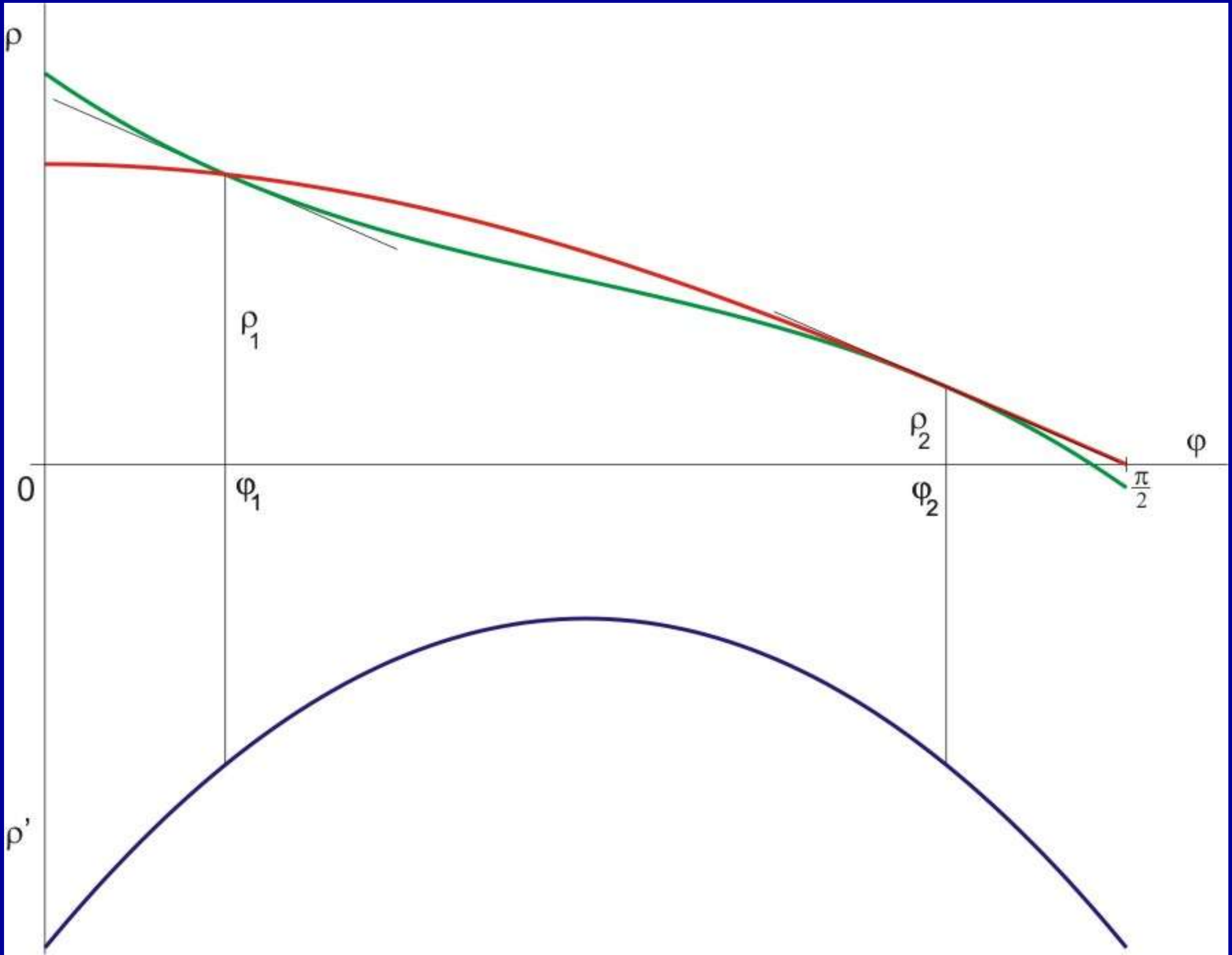
7. Azimuthal Projection with Two Standard Parallels (cont.)



8. Azimuthal Projection with Three Standard Parallels

$\varphi = \varphi_1$ $\varphi = \varphi_2$ $\varphi = \varphi_3$

$\rho(\varphi_1) = R\cos\varphi_1$ $\rho(\varphi_2) = R\cos\varphi_2$ $\rho(\varphi_3) = R\cos\varphi_3$

$\rho'(\varphi_1) = -R$ $\rho'(\varphi_2) = -R$ $\rho'(\varphi_3) = -R$

We need a function $\rho = \rho(\varphi)$

passing through three given points in a given direction

$$\rho = a\varphi^5 + b\varphi^4 + c\varphi^3 + d\varphi^2 + e\varphi + f$$

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8. Azimuthal Projection with Three Standard Parallels (cont.)

Example

$R = 1$ $\varphi_1 = 15°\text{N}$ $\varphi_2 = 45°\text{N}$ $\varphi_3 = 75°\text{N}$

$a = -6.48131$

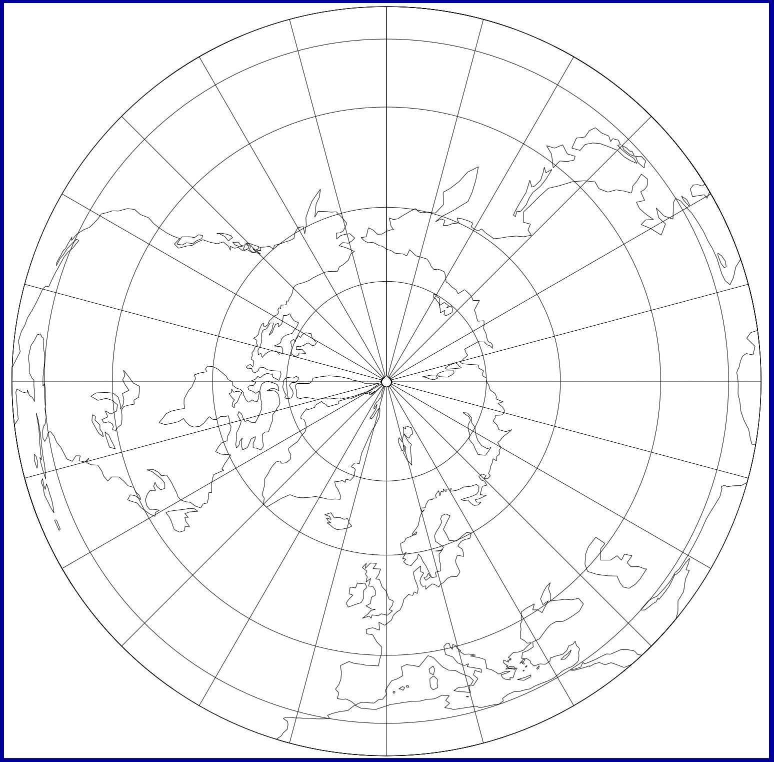
$b = 26.71247$

$c = -40.97822$

$d = 28.39621$

$e = -9.20743$

$f = 2.04796$



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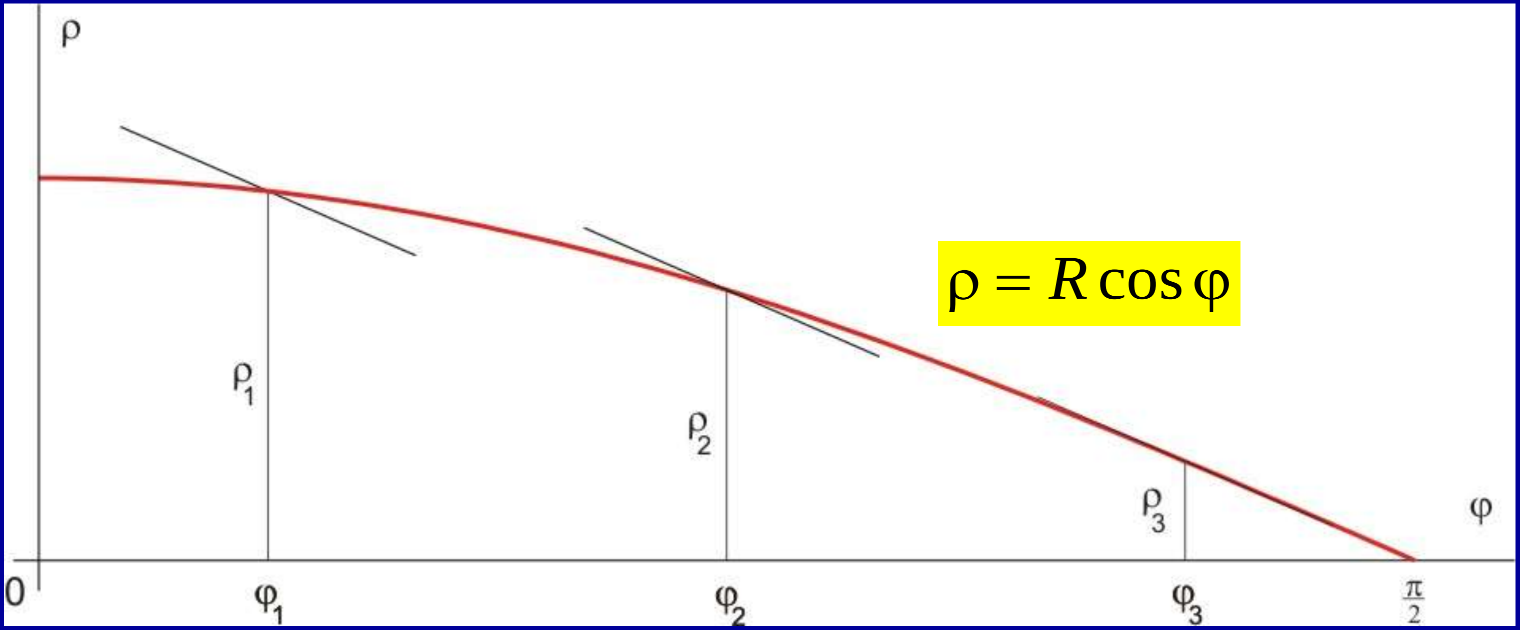
8. Azimuthal Projection with Three Standard Parallels (cont.)



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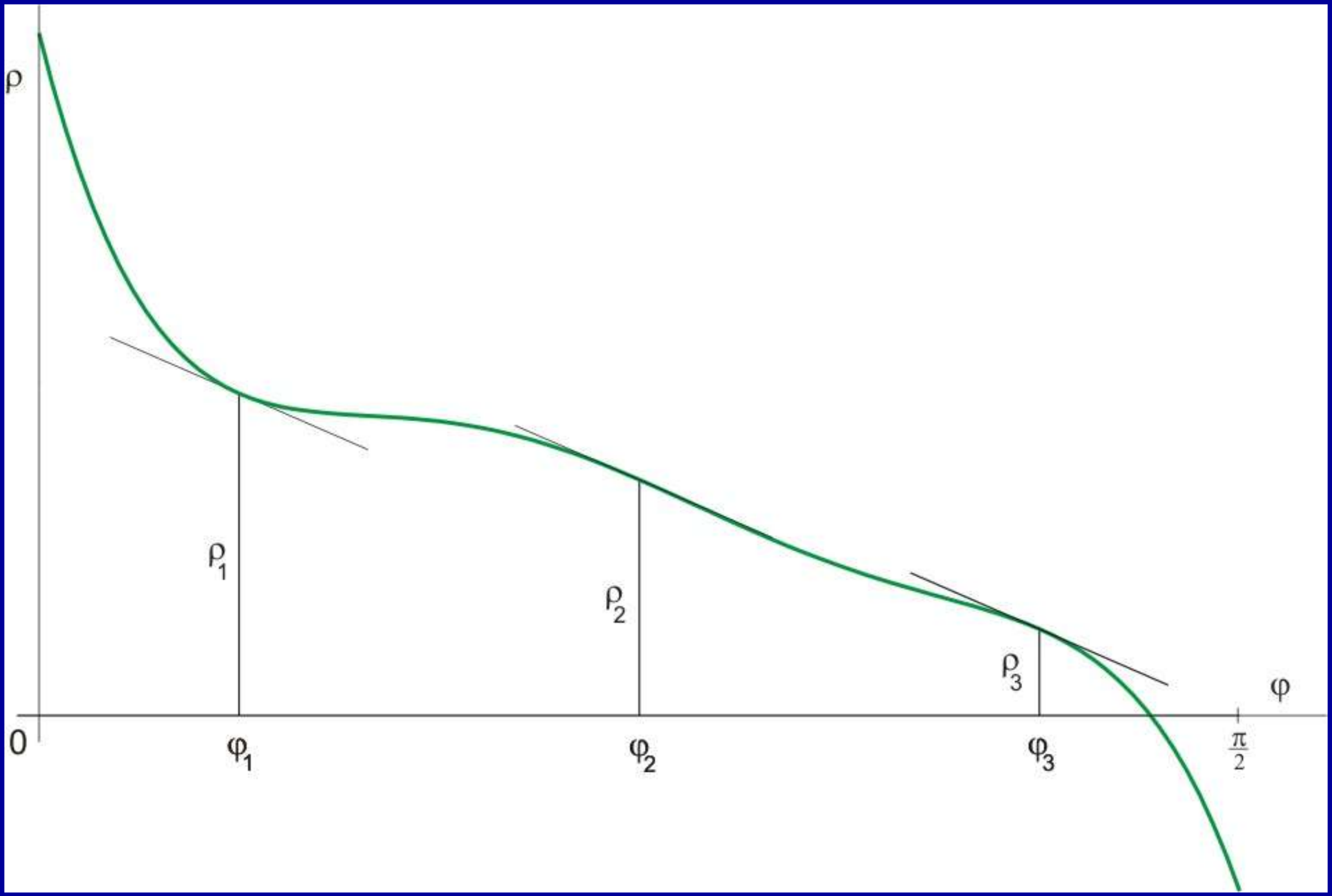
8. Azimuthal Projection with Three Standard Parallels (cont.)



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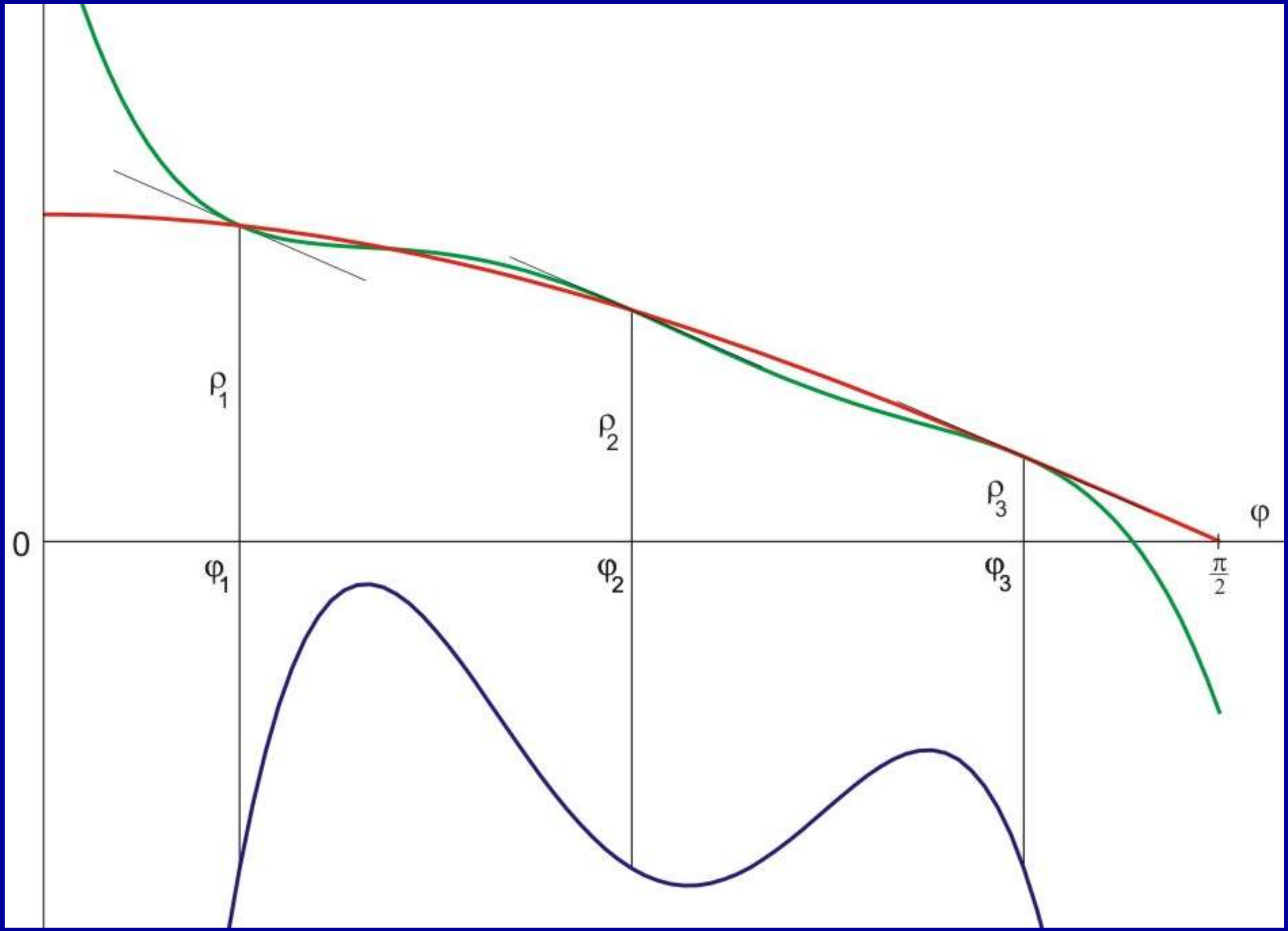
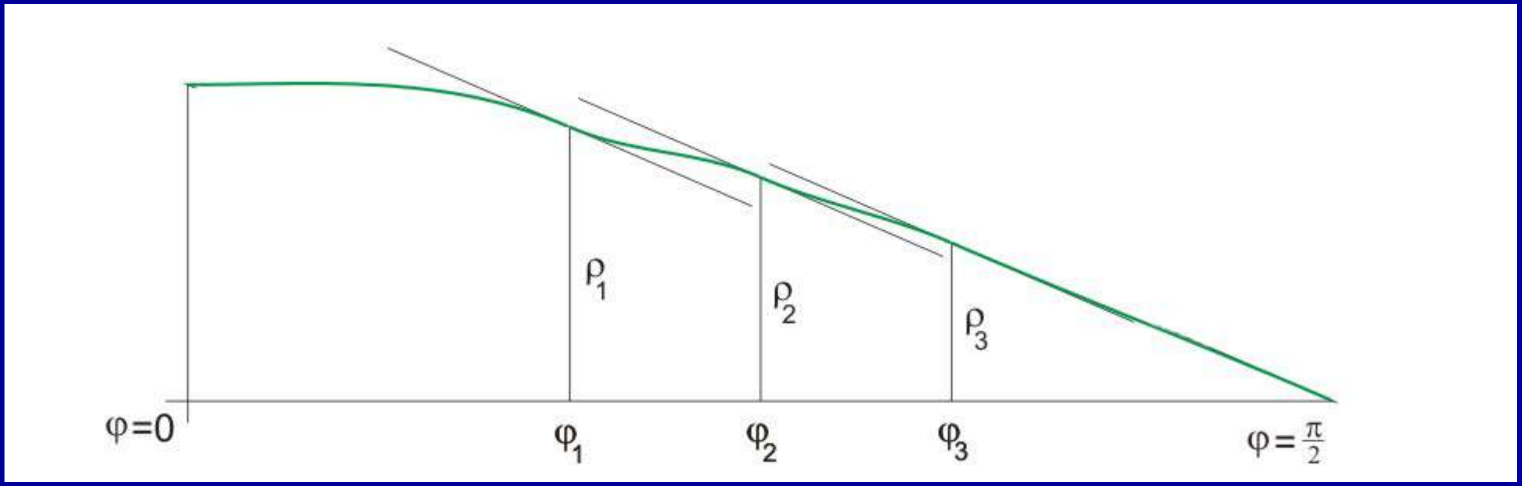
8. Azimuthal Projection with Three Standard Parallels (cont.)



8. Azimuthal Projection with Three Standard Parallels (cont.)



Hermite spline interpolation

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Conclusions

It was demonstrated the existence of azimuthal projections with more than one standard parallel.

It follows that relating the projection plane to a projecting sphere generally does not make much sense

It is not possible to have a plane intersecting a sphere in two, three or more concentric circles.

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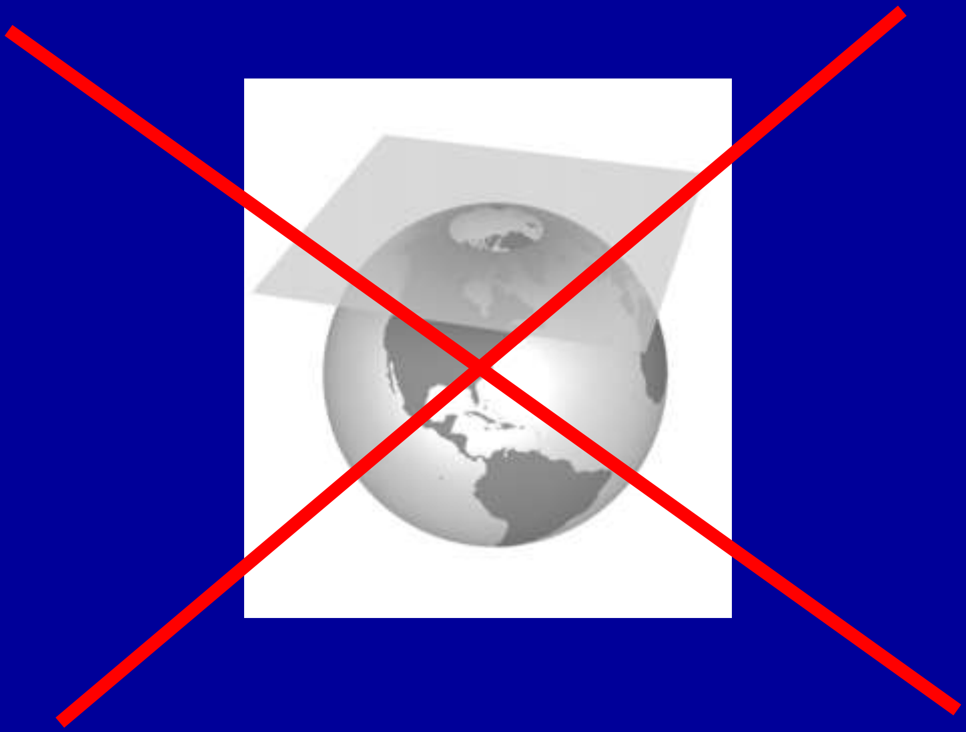

Thank you!

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