Application of 2D Bisection Method for the Inverse Winkel Tripel Projection

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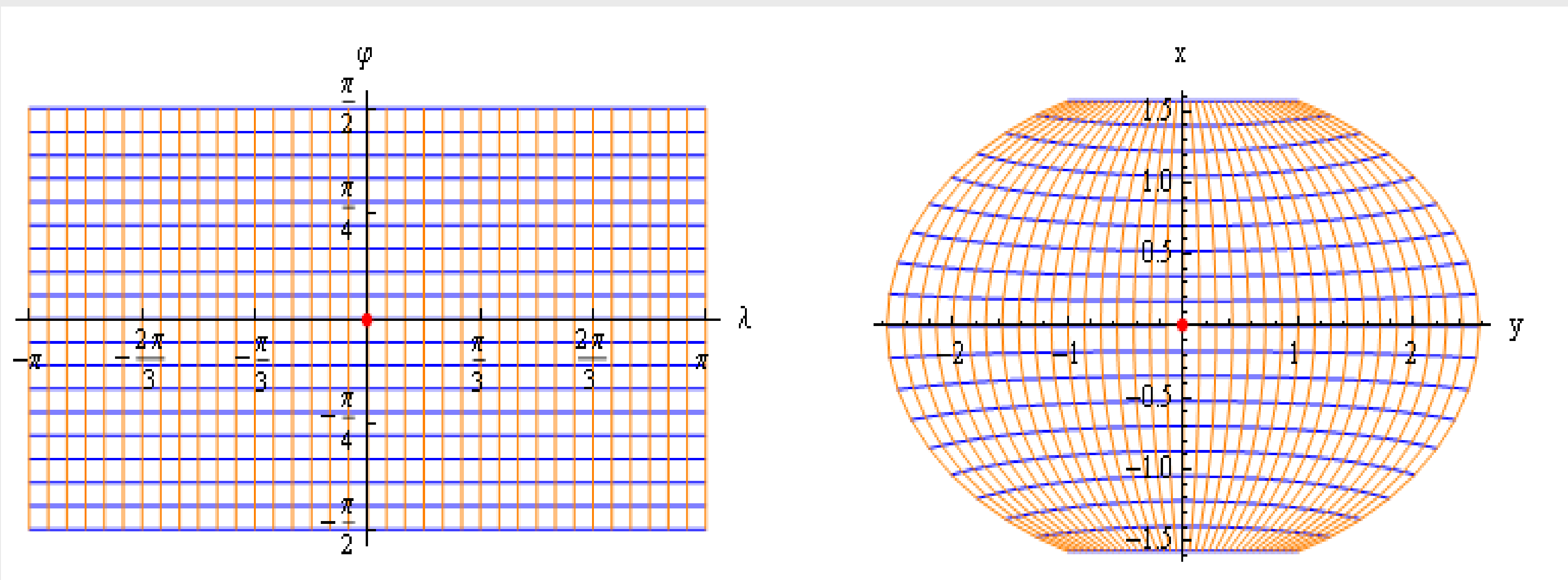
Winkel Tripel - forward

plane coordinates $(\boldsymbol{x}, \boldsymbol{y})$:

$$x(\varphi,\lambda) = \frac{R}{2}\left(\varphi + \frac{z\ \sin\varphi}{\sqrt{1-\left(\cos\frac{\lambda}{2}\cos\varphi\right)^2}}\right),\ y(\varphi,\lambda) = \frac{R}{2}\left(\lambda\cos\widetilde{\varphi_0} + \frac{2\,z\cos\varphi\sin\frac{\lambda}{2}}{\sqrt{1-\left(\cos\frac{\lambda}{2}\cos\varphi\right)^2}}\right)$$

where $z = \arccos\left(\cos\frac{\lambda}{2}\cos\varphi\right)$

- singular point $(\varphi = 0, \lambda = 0)$!



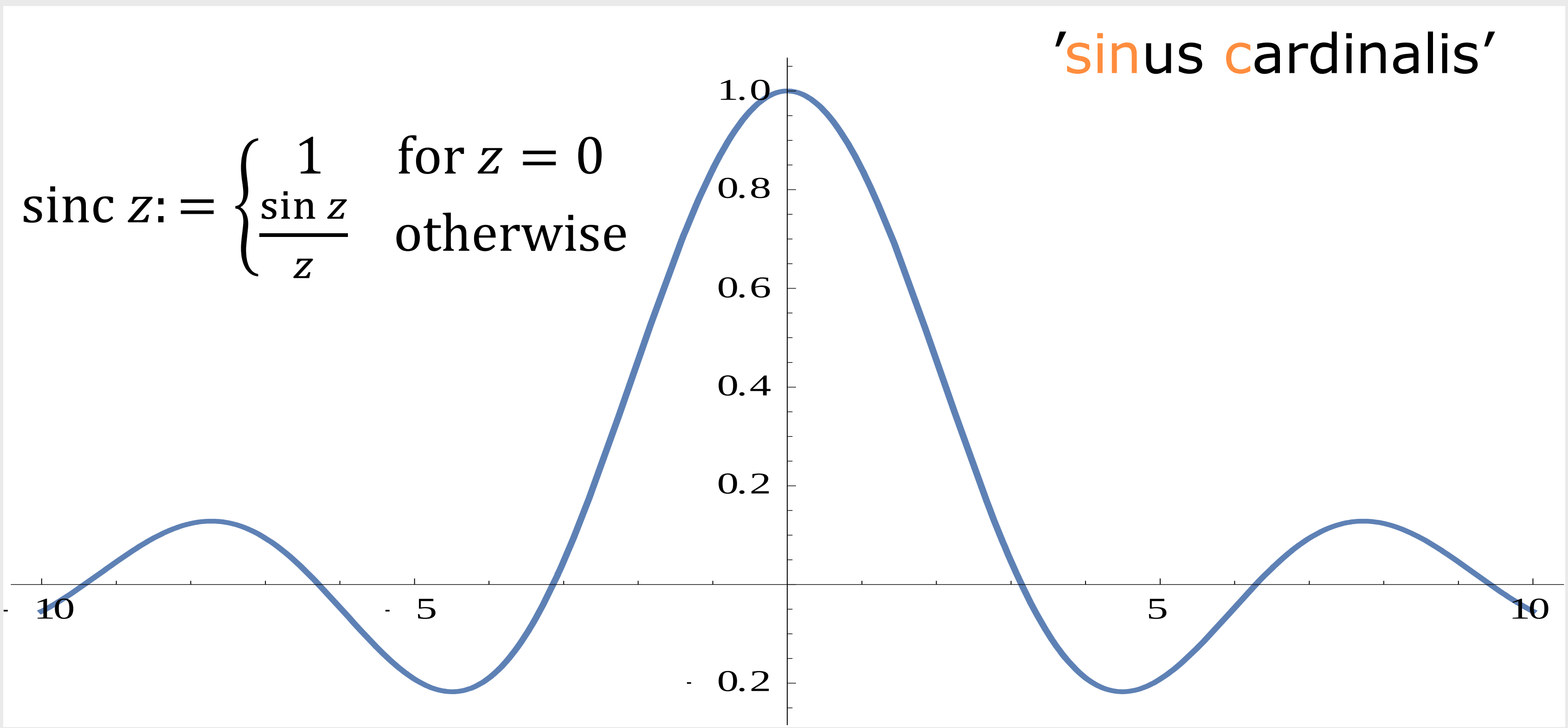
$$\widetilde{\varphi_0} = \arccos\frac{2}{\pi} = 50°27'35.1945'',\quad R = 6370\ \text{km}$$

Winkel Tripel - forward

$$x(\varphi,\lambda) := \frac{R}{2}\left(\varphi + \frac{\sin\varphi}{\operatorname{sinc} z}\right),\ y(\varphi,\lambda) := \frac{R}{2}\left(\lambda \cos\widetilde{\varphi_0} + \frac{2\cos\varphi \sin\frac{\lambda}{2}}{\operatorname{sinc} z}\right)$$

- continuous on the whole domain $\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \times [-\pi,\pi]$!

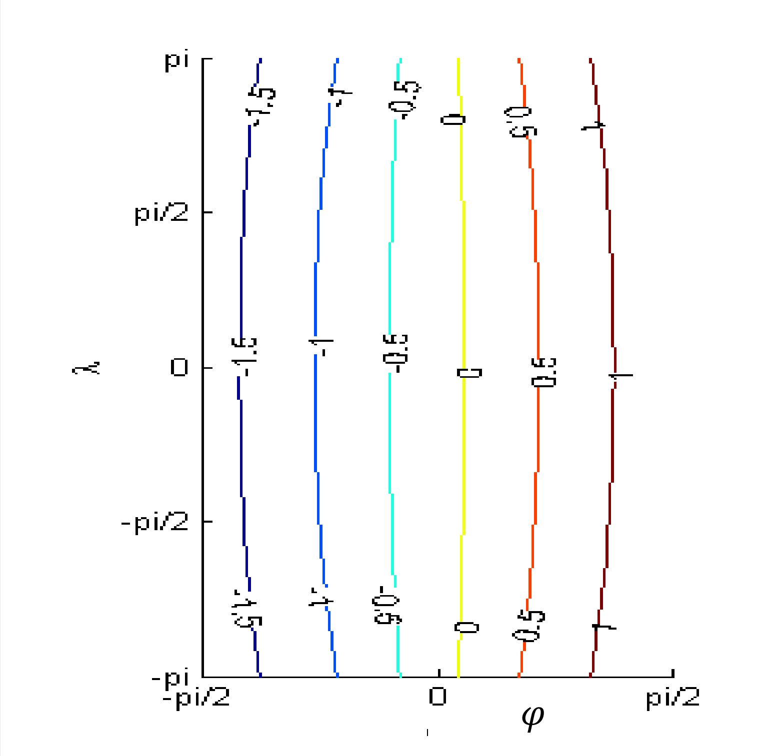
$$\operatorname{sinc} z := \begin{cases} 1 & \text{for } z = 0 \\ \frac{\sin z}{z} & \text{otherwise} \end{cases}$$

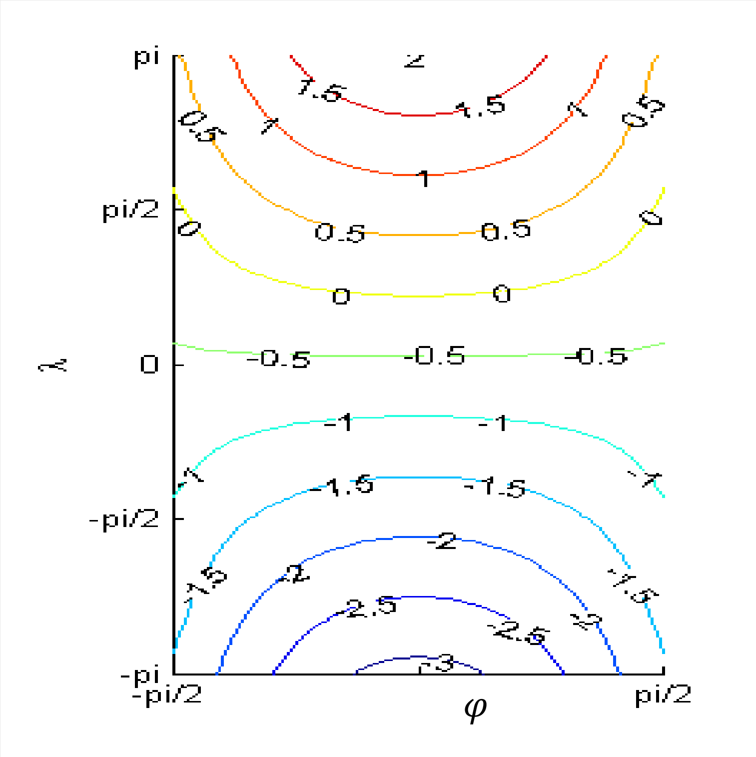


Winkel Tripel - inverse

geographic coordinates (φ, λ): $G_1(\varphi,\lambda) = 0,\ G_2(\varphi,\lambda) = 0$

- nonlinear system!





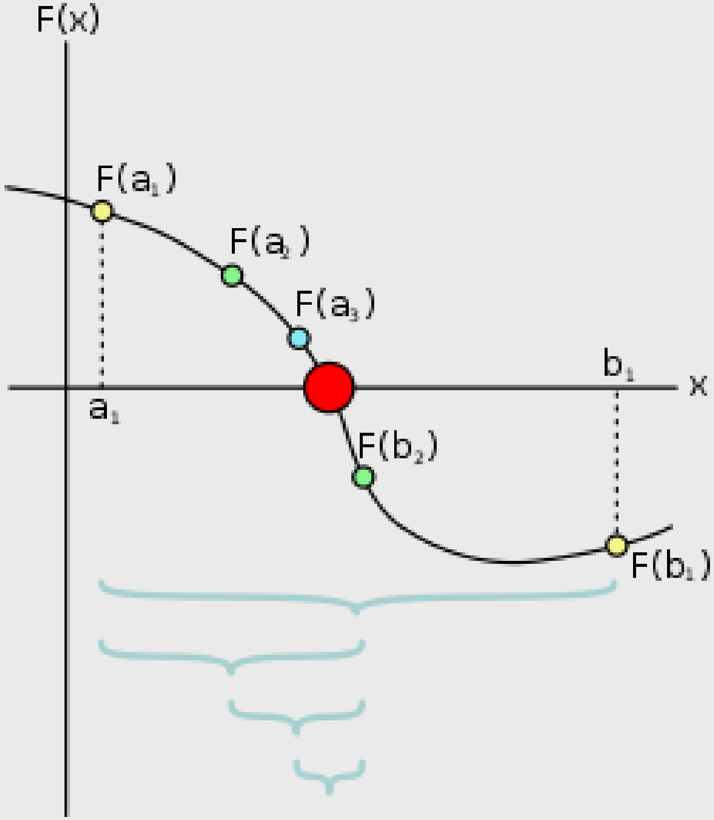
$$G_1(\varphi,\lambda) := \frac{R}{2}\left(\varphi + \frac{\sin\varphi}{\operatorname{sinc} z}\right) - x$$

$$G_2(\varphi,\lambda) := \frac{R}{2}\left(\lambda\cos\widetilde{\varphi_0} + \frac{2\cos\varphi\sin\frac{\lambda}{2}}{\operatorname{sinc} z}\right) - y$$

1D Bisection method - algorithm

Find solution $x^* \epsilon (a, b)$ of the nonlinear equation $\boxed{\mathrm{F}(x) = 0}$

$\mathrm{F}(x)$ is continous on $[a, b]$ and it satisfies $\mathrm{F}(a) * \mathrm{F}(b) < 0$!



Step 1 $x = \frac{a+b}{2}$

Step 2 *if* $F(a) * F(x) < 0$:
$b = x$
else
$a = x$

Step 3 *if* $b - a > TOL$:
go to **Step 1**

TOL - prescribed tolerance

This graph is retrieved from http://en.wikipedia.org/wiki/Bisection_method

2D Bisection method - algorithm

Step 1 $\varphi_{2,min} = -\frac{\pi}{2} - \varepsilon,\ \ \varphi_{2,max} = \frac{\pi}{2} + \varepsilon,\ \ \lambda_{2,min} = -\pi - \varepsilon,\ \ \lambda_{2,max} = \pi + \varepsilon$

Step 2 $\lambda = \frac{\lambda_{2,min} + \lambda_{2,max}}{2}$

Step 3

find solution $\varphi_{1,min} \epsilon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ of the nonlinear equation $G_1(\varphi, \lambda_{2,min}) = 0$

and find solution $\varphi_{1,pol} \epsilon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ of the nonlinear equation $G_1(\varphi, \lambda_{2,pol}) = 0$

Step 4 $\varphi = \frac{\varphi_{1,min} + \varphi_{1,pol}}{2}$

Step 5 *if* $G_2\left(\varphi_{1,min}, \lambda_{2,min}\right) * G_2\left(\varphi_{1,pol}, \lambda_{2,pol}\right) < 0$:

$\lambda_{2,max} = \lambda_{2,pol}$

else

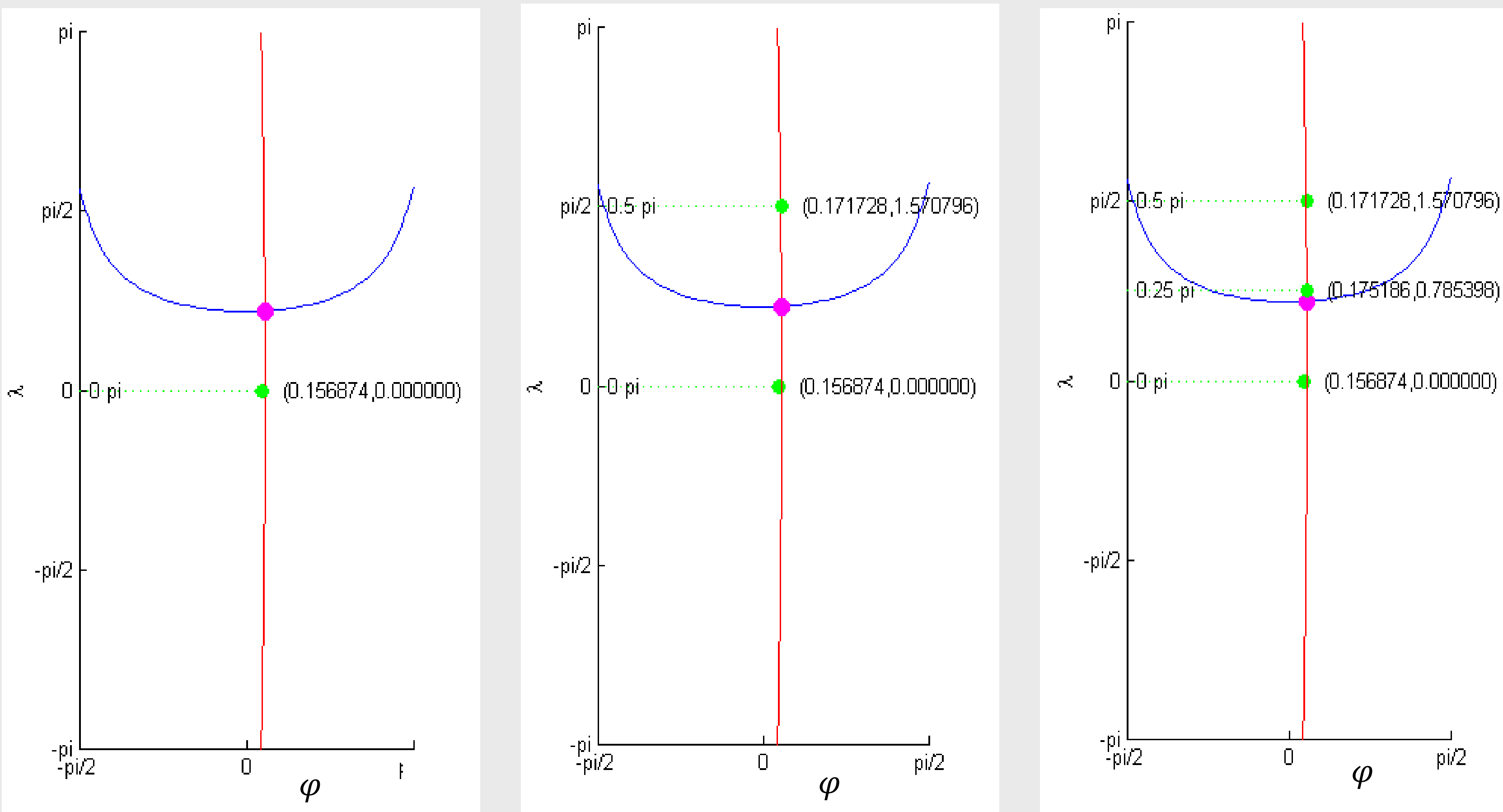
$\lambda_{2,min} = \lambda_{2,pol}$

Step 6 *if* $\lambda_{2,max} - \lambda_{2,min} > TOL$:

go to **Step 2**

2D Bisection method – example 1

For $(x, y) = (0.17632, 0.56768)$: first three approximations



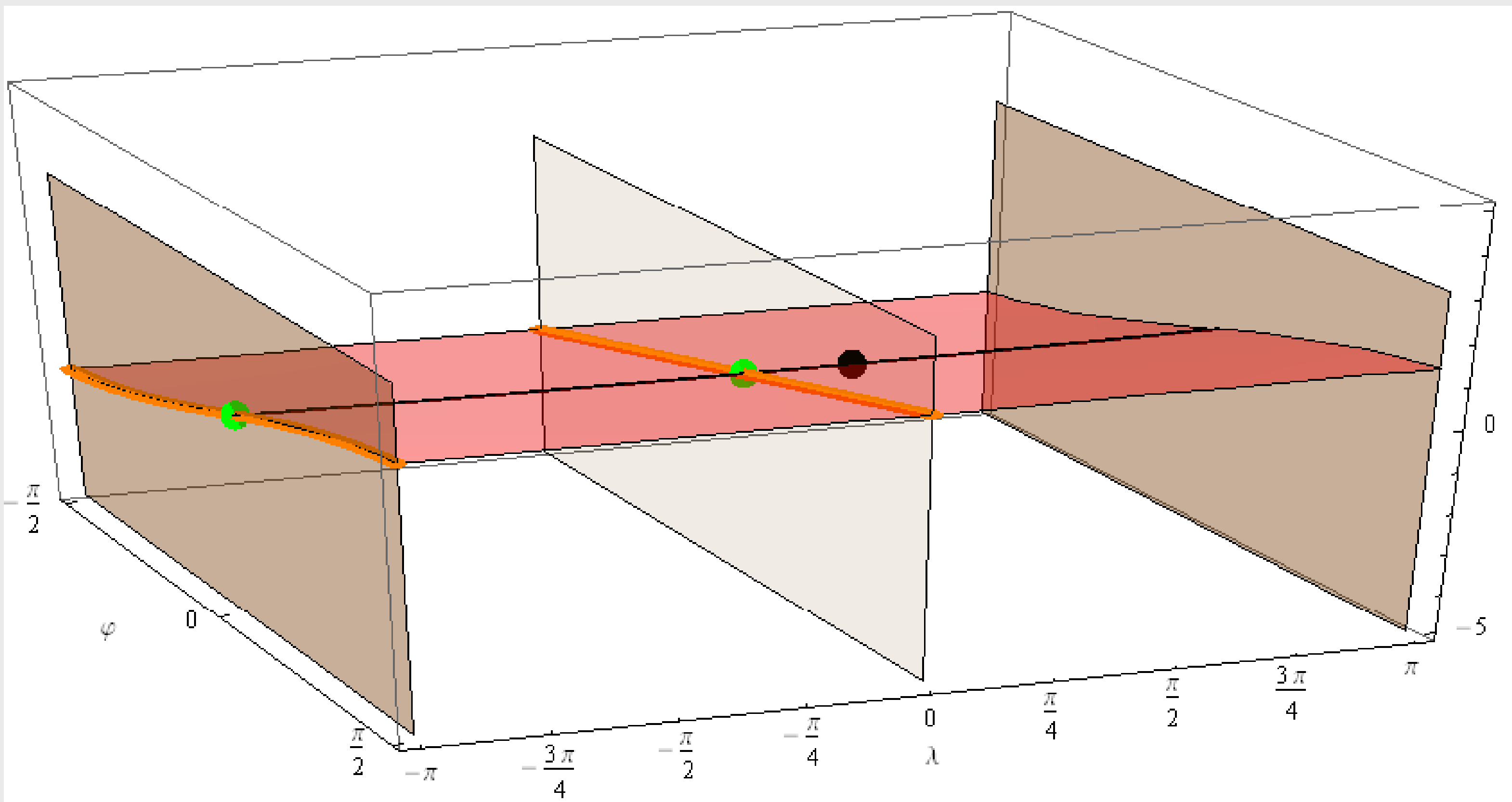
Exact solution is $(\varphi^*, \lambda^*) = (10°, 40°) \approx (0.17453\ \mathrm{rad}, 0.69813\ \mathrm{rad})$

$R = 1,\ \widetilde{\varphi_0} = \arccos\frac{2}{\pi}$

2D Bisection method – example 1

Step 3 solutions of $G_1(\varphi, \lambda_{2,min}) = 0$ and $G_1(\varphi, \lambda_{2,pol}) = 0$

first interval $[-\pi, \pi]$, $\lambda_{2,min} = -\pi - \varepsilon$, $\lambda_{2,pol} = 0$

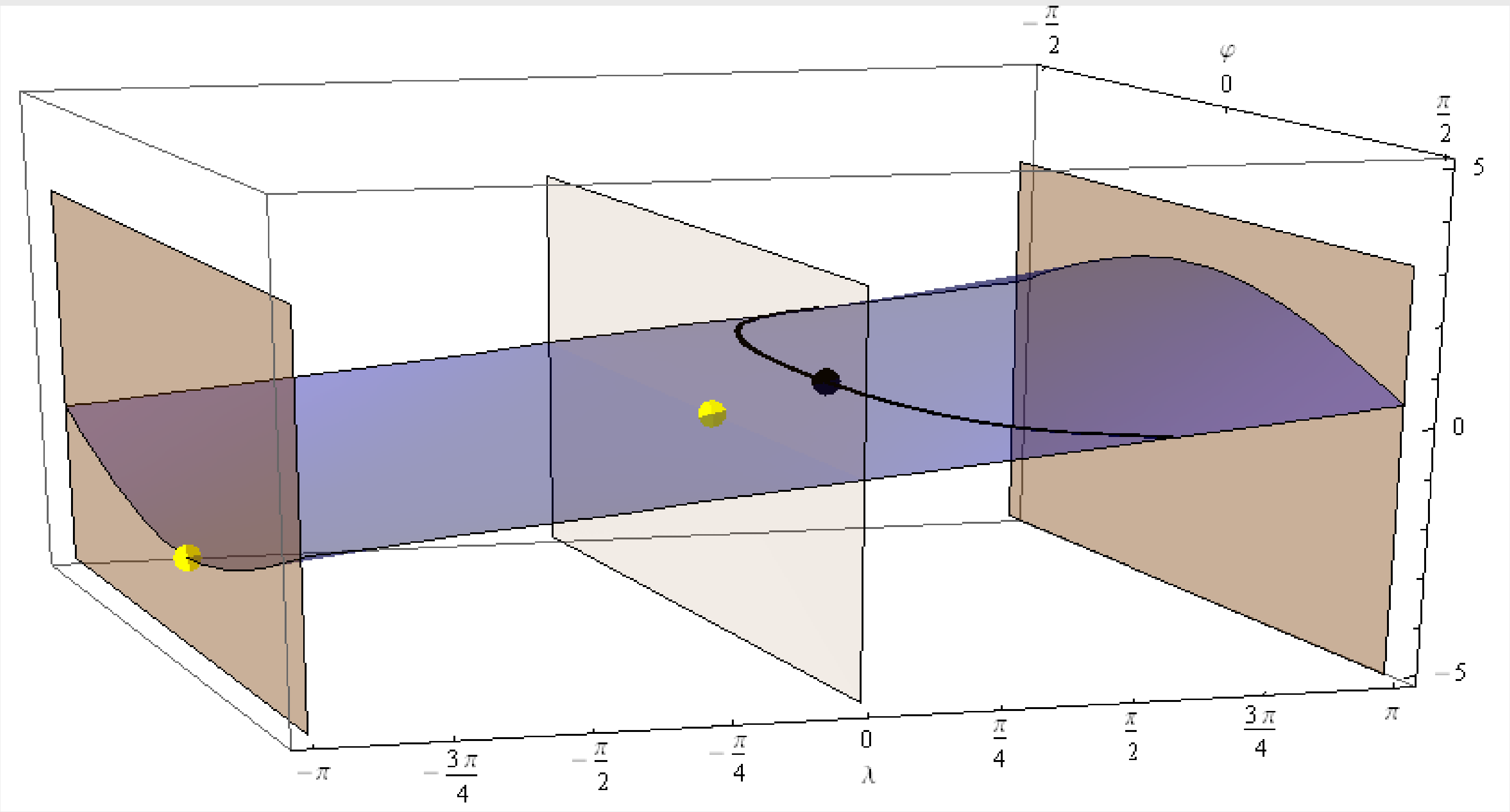


green points … G_1 for $(\varphi_{1,min}, \lambda_{2,min})$, $(\varphi_{1,pol}, \lambda_{2,pol})$

2D Bisection method – example 1

Step 5 comparison of $G_2\left(\varphi_{1,min}, \lambda_{2,min}\right)$ and $G_2\left(\varphi_{1,pol}, \lambda_{2,pol}\right)$

first interval $[-\pi, \pi]$, $\lambda_{2,min} = -\pi - \varepsilon$, $\lambda_{2,pol} = 0$



yellow points … G_2 for $\left(\varphi_{1,min}, \lambda_{2,min}\right)$, $\left(\varphi_{1,pol}, \lambda_{2,pol}\right)$

2D Bisection method – example 1

Iterations (in the main loop for λ) with $TOL = 10^{-12}$

k	φ_k(rad)	$\lvert\varphi - \varphi_k\rvert$	λ_k(rad)	$\lvert\lambda - \lambda_k\rvert$
1	0.156873907371925	0.017659017827508	0.000000000000000	0.698131700797732
2	0.171728211859678	0.002804713339755	1.570796331794897	0.872664630997165
3	0.175186281268449	0.000653356069016	0.785398165897448	0.087266465099716
4	0.176034385283641	0.001501460084208	0.392699082948724	0.305432617849008
5	0.175400570994769	0.000867645795336	0.589048624423086	0.109083076374646
6	0.174818206646106	0.000285281446673	0.687223395160267	0.010908305637465
7	0.174460058240507	0.000072866958926	0.736310780528858	0.038179079731126
8	0.174525363860450	0.000007561338983	0.711767087844563	0.013635387046831
9	0.174557172439893	0.000024247240460	0.699495241502415	0.001363540704683
10	0.174572865883688	0.000039940684255	0.693359318331341	0.004772382466391
11	0.174549483766723	0.000016558567290	0.696427279916878	0.001704420880854
…				
35	0.174532925200101	0.000000000000668	0.698131700723985	0.000000000073746
36	0.174532925199387	0.000000000000046	0.698131700815418	0.000000000017686
37	0.174532925199744	0.000000000000311	0.698131700769702	0.000000000028030
38	0.174532925199744	0.000000000000311	0.698131700792560	0.000000000005172
39	0.174532925199744	0.000000000000311	0.698131700803989	0.000000000006257
40	0.174532925199744	0.000000000000311	0.698131700798274	0.000000000000542
41	0.174532925199744	0.000000000000311	0.698131700795417	0.000000000002315
42	0.174532925199744	0.000000000000311	0.698131700796846	0.000000000000886
43	0.174532925199744	0.000000000000311	0.698131700797560	0.000000000000172
	φ (rad)= 0.174532925199433		λ(rad) = 0.698131700797732	

2D Bisection method – example 2

Results for some points with $TOL = 10^{-12}$, $\varepsilon = 10^{-8}$

φ (rad) / λ (rad)	φ_k(rad) / λ_k(rad)	$\lvert\varphi - \varphi_k\rvert$ / $\lvert\lambda - \lambda_k\rvert$	x (km) / y (km)	$\varphi(°)$ / $\lambda(°)$
1.012290966157	1.012290966157	1.97176e-013	6599.949178610303	**58**
1.274090353956	1.274090353956	4.17888e-013	5092.493117785527	**73**
1.012290966157	1.012290966157	1.97176e-013	6604.225078196277	**58**
1.291543646476	1.291543646476	3.33067e-015	5160.592147601173	**74**
1.012290966157	1.012290966157	1.97176e-013	6608.564212234337	**58**
1.308996938996	1.308996938995	4.11893e-013	5228.620172388051	**75**
1.012290966157	1.012290966157	1.97176e-013	6612.966784615432	**58**
1.326450231516	1.326450231515	8.26894e-013	5296.576009916544	**76**
1.012290966157	1.012290966157	1.97176e-013	6617.433001829236	**58**
1.343903524036	1.343903524036	1.86740e-013	5364.458468395563	**77**
1.012290966157	1.012290966157	1.97176e-013	6621.963072952889	**58**
1.361356816556	1.361356816555	2.28262e-013	5432.266346305216	**78**

$R = 6370$ km, $\widetilde{\varphi_0} = 50°28'$

2D Bisection method

Why does it work?

- The functions $G_1(\varphi,\lambda)$ and $G_2(\varphi,\lambda)$ are continous on $\left[-\frac{\pi}{2}-\varepsilon,\frac{\pi}{2}+\varepsilon\right]\times[-\pi-\varepsilon,\pi+\varepsilon]$

  ε – small constant

- The function $G_1(\varphi) \coloneqq G_1(\varphi,\lambda_0 = \text{const})$ is strictly increasing and $G_1\left(-\frac{\pi}{2}-\varepsilon,\lambda_0\right) * G_1\left(\frac{\pi}{2}+\varepsilon,\lambda_0\right) < 0$

  $\Rightarrow$ function $G_1(\varphi)$ has only one zero in $\left(-\frac{\pi}{2}-\varepsilon,\frac{\pi}{2}+\varepsilon\right)$

  $\Rightarrow$ 1D bisection method could be applied in **Step 3**

- The function $G_2(\lambda) \coloneqq G_2(\varphi_0 = \text{const},\lambda)$ is strictly increasing and $G_2(\varphi_0,-\pi-\varepsilon) * G_2(\varphi_0,\pi+\varepsilon) < 0$

  $\Rightarrow$ function $G_2(\lambda)$ has only one zero in $(-\pi-\varepsilon,\pi+\varepsilon)$

  $\Rightarrow$ 1D bisection method could be applied in the main loop

Bisection method - accuracy

1D bisection

- After k iterations, the error is less then $\frac{b-a}{2^k}$
- The number of iterations k needed to achieve a specified accuracy TOL is known in advance:

$$k > \log_2 \frac{b-a}{TOL} \Longrightarrow k = \left\lceil \frac{\log(b-a) - \log TOL}{\log 2} \right\rceil$$

$k = \lceil 41.51 \ldots \rceil = 42$ for $a = -\frac{\pi}{2} - \varepsilon,\ b = \frac{\pi}{2} + \varepsilon,\ TOL = 10^{-12}, \varepsilon = 10^{-8}$

$k = \lceil 42.51 \ldots \rceil = 43$ for $a = -\pi - \varepsilon,\ b = \pi + \varepsilon,\ TOL = 10^{-12}, \varepsilon = 10^{-8}$

2D bisection for the inverse Winkel Tripel projection

- The interval halvings needed to achieve the accuracy TOL

$$2 * \left\lceil \frac{\log(2\pi + 2\varepsilon) - \log TOL}{\log 2} \right\rceil * \left\lceil \frac{\log(\pi + 2\varepsilon) - \log TOL}{\log 2} \right\rceil$$

$43 * 2 * 42 = 3612$ interval halvings for $TOL = 10^{-12}, \varepsilon = 10^{-8}$

Our future plans

- Inverse mapping transformations
- Numerical solutions of nonlinear problems in cartography

Thank You!